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Generalized geometric phase of a classical oscillator

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Abstract
The equation of a linear oscillator with adiabatically varying eigenfrequency $\omega(\varepsilon t)$ ($\varepsilon \ll 1$ is the adiabaticity parameter) is considered. The asymptotic solutions to the equation have been obtained to terms of order ε^3. It is shown that imaginary terms of order ε^2 form a generalized geometric phase determined by the geometry of the system's contour in the plane (ω, ω'). The real terms of orders ε and ε^3, as predicted (Bliokh K Yu 2002 *J. Math. Phys.* **43** 5624), do not form geometric amplitudes but are responsible for local relationships between the solution amplitudes and the parameters, that is, for adiabatic invariants.

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1. Introduction

This paper develops the ideas of papers [1–3]. In particular, in [3] the notion of the *generalized geometric phase* has been introduced; its real part (generalized geometric amplitude) has been proved to be zero in Hamiltonian oscillatory systems. This fact is closely related to the strong stability and quantizability of Hamiltonian systems. The generalized geometric phase is, in fact, the analogue of Berry's geometric phase or Hannay's angle [4–7], but it is constructed, however, not in the parameter space, but in the *generalized parameter space* [3], which includes not only dimensions of parameters, but dimensions of their derivatives as well. Below, we will give some particular examples supporting general theorems proved previously and will demonstrate the initiation of the generalized geometric phase in a harmonic oscillator with a slowly varying eigenfrequency.

Berry's phase or Hannay's angle by no means arises in every system; independent variation of several real parameters provides the necessary condition for its initiation in the case of ordinary linear differential equations. Conversely, the generalized geometric phase can appear even in a very simple system (classical oscillator) under adiabatic changes of the single parameter—oscillator eigenfrequency. Note that the latter effect corresponds to higher orders of approximation in a small adiabaticity parameter.

2. Basic calculations

Consider the equation for a classical linear oscillator

$$x'' + \omega^2(\varepsilon t)x = 0 \tag{1}$$

where the prime stands for differentiation with respect to time t and $\varepsilon \ll 1$ is a small adiabaticity parameter. Variations of the real eigenfrequency ω are assumed to be finite and not closely approaching the turning point $\omega = 0$. Then the asymptotic solutions to equation (1) can be constructed to an arbitrary accuracy in ε. The Neistadt method of successive diagonalizations [3] permits this to be done and shows that these solutions can always be written in the form of an exponential function of some complex phase[1] (see also [8]). With such a representation, the method of successive approximations for the exponent is more convenient for calculations and will lead to the desired result. The asymptotic solutions to equation (1) with an accuracy of ε^3 are

$$x = x(0)\exp\left\{\int_0^t \left[\pm \mathrm{i}\omega - \frac{\omega'}{2\omega} \pm \mathrm{i}\frac{3\omega'^2}{8\omega^3} \mp \mathrm{i}\frac{\omega''}{4\omega^2} + \frac{3\omega'^3}{4\omega^5} - \frac{3\omega'\omega''}{4\omega^4} + \frac{\omega'''}{8\omega^3} + O(\varepsilon^4)\right]\mathrm{d}\tau\right\}. \tag{2}$$

By substituting (2) into (1) we can easily verify that the above solution is correct.

The first term in the integrand in (2) represents a regular dynamic phase. The other terms can be represented in geometric form. To this end, let us introduce a generalized three-dimensional space of the parameter ω: $\vec{M} = (\omega, \omega', \omega'')$ [3]. Solutions (2) can be written as

$$x = x(0)\exp\left\{\pm \mathrm{i}\int_0^t \omega\,\mathrm{d}\tau + \int_L \vec{F}\,\mathrm{d}\vec{M}\right\} + O(\varepsilon^4 t). \tag{3}$$

Here the second integral is taken along the trajectory L, which is the trajectory of the representative point of the system in $\vec{M}$-space. The field $\vec{F}(\vec{M})$ is equal to

$$\vec{F} = \left(-\frac{1}{2\omega} \pm \mathrm{i}\frac{3\omega'}{8\omega^3} + \frac{3\omega'^2}{4\omega^5} - \frac{3\omega''}{8\omega^4}, \mp \mathrm{i}\frac{1}{4\omega^2} - \frac{3\omega'}{8\omega^4}, \frac{1}{8\omega^3}\right). \tag{4}$$

Note that the next to the last term in the integrand in (2) could be assigned both to the first component of the field $\vec{F}$ through the substitution $\omega'\omega''\,\mathrm{d}t = \omega''\,\mathrm{d}\omega$ and to the second component through the substitution $\omega'\omega''\,\mathrm{d}t = \omega'\,\mathrm{d}\omega'$. In (4), an intermediate variant has been chosen: $\omega'\omega''\,\mathrm{d}t = (\omega''\,\mathrm{d}\omega + \omega'\,\mathrm{d}\omega')/2$. The meaning of such a representation is discussed below.

It is known that nonlocal terms (generalized geometric phases in solution (3)) appear because the field $\vec{F}$ is nonpotential [1, 3]. As for the potential part of the field, it can be integrated to give only local dependence on current values of system parameters. To isolate the nonpotential component of the field $\vec{F}$, we have to calculate its curl in $\vec{M}$-space. As a

[1] Even pre-exponential factors arising as a result of substitution of variables in successive diagonalizations are local values (functions) of the parameters and their derivatives and can always be transferred into the integrand to the exponent.

result we arrive at

$$
\begin{aligned}
(\operatorname{rot}\vec{F})_1 &= \frac{\partial F_3}{\partial \omega'} - \frac{\partial F_2}{\partial \omega''} = 0 \\
(\operatorname{rot}\vec{F})_2 &= \frac{\partial F_1}{\partial \omega''} - \frac{\partial F_3}{\partial \omega} = 0 \\
(\operatorname{rot}\vec{F})_3 &= \frac{\partial F_2}{\partial \omega} - \frac{\partial F_1}{\partial \omega'} = \pm\frac{\mathrm{i}}{8\omega^3}.
\end{aligned}
\tag{5}
$$

The third curl component is nonzero, hence the field is nonpotential and the solution will possess a generalized geometric phase. At the same time, it will not have a generalized geometric amplitude (as predicted in [3] for oscillatory Hamiltonian systems), since $\operatorname{Re}(\operatorname{rot}\vec{F}) = 0$. It can easily be shown that had we assigned the next to last term of the integrand in (2) completely to the first or the second field component, the function rot $\vec{F}$ would have had nonzero real parts in the second and the third components. An integral of these parts along the closed trajectory of the representative point of the system in $\vec{M}$-space would be, however, identically equal to zero and they would not produce the generalized geometric amplitudes[2].

By comparing (5) with (4), we can isolate the nonpotential part of the field $\vec{F}$

$$
\vec{F}^{(c)} = \left(\mp\mathrm{i}\frac{\omega'}{8\omega^3}, 0, 0\right). \tag{6}
$$

The potential component has the form $\vec{F}^{(p)} = \operatorname{grad}\varphi$, where $\varphi(\vec{M})$ is the scalar potential, which is equal, in this case

$$
\varphi = -\frac{1}{2}\ln\omega \mp \mathrm{i}\frac{\omega'}{4\omega^2} - \frac{3\omega'^2}{16\omega^4} + \frac{\omega''}{8\omega^3}. \tag{7}
$$

Upon integrating the potential component $\vec{F}^{(p)}$, we obtain solution (3) in the form [3]

$$
x = x(0)\exp\left\{\pm\mathrm{i}\int_0^t \omega\,\mathrm{d}\tau + \varphi(t) - \varphi(0) + \int_L \vec{F}^{(c)}\,\mathrm{d}\vec{M}\right\} + O(\varepsilon^4 t) \tag{8}
$$

or, after substituting (6) and (7) into (8), we have

$$
\begin{aligned}
x = x(0)\exp\Bigg\{&\pm\mathrm{i}\int_0^t \omega\,\mathrm{d}\tau + \left[-\frac{1}{2}\ln\omega \mp \mathrm{i}\frac{\omega'}{4\omega^2} - \frac{3\omega'^2}{16\omega^4} + \frac{\omega''}{8\omega^3}\right]\Bigg|_0^t \\
&\mp \mathrm{i}\int_L \frac{\omega'}{8\omega^3}\,\mathrm{d}\omega\Bigg\} + O(\varepsilon^4 t).
\end{aligned}
\tag{9}
$$

The first term in (9) is the dynamic phase as before. The terms in square brackets are local and, consequently, cannot grow infinitely. They are equal to zero under cyclic changes (when ω and its derivative return to their initial values). The first term in the square brackets can be represented as the pre-exponential factor $\sqrt{\omega(0)/\omega(\varepsilon t)}$. It is responsible for constructing the known adiabatic invariant of the oscillator. It is readily seen that owing to this term the value $I = |x(t)|^2\omega(t) = \text{const} + O(\varepsilon^2 t)$ is conserved in the first approximation in ε. The next term in the square brackets is purely imaginary and always small (of order ε); it corrects the

[2] This is due to the fact that $\vec{M}$-space is not a regular space with independent dimensions. Its dimensions characterize a single function (on this topic, see in [3]).

current phase of the solutions. The two last terms in the square brackets are also always small (of order ε^2); they represent corrections to the adiabatic invariant. In a third approximation in ε, the adiabatic invariant can be written as

$$I = |x|^2\omega \exp\left[\frac{3\omega'^2}{8\omega^4} - \frac{\omega''}{4\omega^3}\right] = \text{const} + O(\varepsilon^4 t). \tag{10}$$

Now we turn to the last term in exponents (8) and (9). It is the generalized geometric phase [3] of the oscillator

$$\psi = \int_L \vec{F}^{(c)}\,\mathrm{d}\vec{M} = \mp\mathrm{i}\int_L \frac{\omega'}{8\omega^3}\,\mathrm{d}\omega. \tag{11}$$

It can be considered on the plane $\vec{m} = (\omega, \omega')$. The terms of higher orders, which cause the dimension ω'' to appear in previous arguments, were taken into account exclusively to show that the real terms corresponding to these corrections do not give rise to geometric amplitudes.

For the sake of simplicity, first we consider the case of closed trajectories of the representative point of the system in the $\vec{m}$-plane. According to the Stokes theorem, the contour integral in (8) can be reduced to a surface one, i.e. to a flux of rot $\vec{F}^{(c)}$ through the surface S spanned on the contour L:

$$\psi = \int_S \mathrm{rot}\,\vec{F}^{(c)}\vec{n}\,\mathrm{d}s = \mp\mathrm{i}n\int_S \frac{1}{8\omega^3}\,\mathrm{d}\omega\,\mathrm{d}\omega' \tag{12}$$

where $\vec{n}$ is a unit normal to the directed surface S and n is equal to +1 or -1 for the contour L directed anticlockwise or clockwise, respectively.

It is noteworthy how naturally and clearly the functional approach describes the phenomena in question compared to the temporal approach (see [1–3]). Let us compare solution (2) obtained with the help of the temporal approach (where the time integrals are treated) with solutions (6)–(9) of the functional approach (the integrals, being functionals of $\omega(\varepsilon t)$, are considered in the generalized parameter space). It is unlikely that essential and insignificant terms can be separated in the cumbersome formula (2), whereas functional solutions (6)–(11) immediately separate nonlocal and local terms very concisely[3].

3. Example

Let us take a look at the simplest case where the change in ω produces closed contours in the plane $\vec{m} = (\omega, \omega')$: periodically varying $\omega(\varepsilon t)$. Assume that

$$\omega = \omega_0 + \omega_1\cos(\varepsilon t) \tag{13}$$

where $\omega_0 > \omega_1 > 0$. The $\omega(\varepsilon t)$ dependence of this sort corresponds to an ellipse in the $\vec{m}$-plane (figure 1).

Next, we calculate the generalized geometric phase (12) gained during one period of $\omega(\varepsilon t)$ or one cycle of the representative point of the system along the contour L. The contour orientation corresponds to $n = -1$; the surface integral over the elliptic surface S (figure 1) is easily reducible to a double integral

$$\psi_0 = \pm\mathrm{i}\int_{\omega_0-\omega_1}^{\omega_0+\omega_1}\mathrm{d}\omega\int_{-\varepsilon\sqrt{\omega_1^2-(\omega-\omega_0)^2}}^{\varepsilon\sqrt{\omega_1^2-(\omega-\omega_0)^2}}\mathrm{d}\omega'\left(\frac{1}{8\omega^3}\right). \tag{14}$$

[3] Note that the fact that the real terms of first, third and fifth orders in complex phase have primitives, i.e., they are local, was obtained by explicit calculations in [8]. But there is no analysis regarding the separation of local and nonlocal terms in the imaginary terms of second and fouth order, because this can be practically done only from the point of view of the geometrical formalism that is used here.

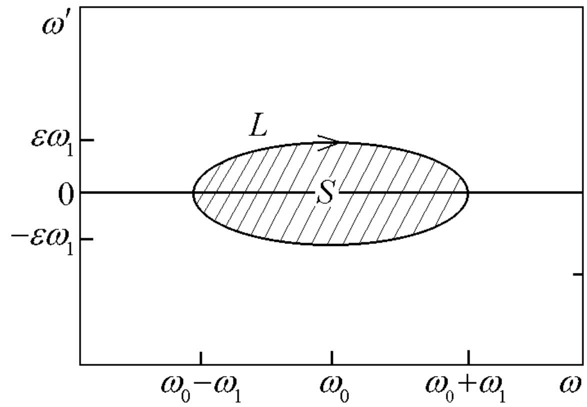

Figure 1. Oriented elliptic contour L and area S corresponding to it in the $\vec{m}$-plane under periodic variations of $\omega(\varepsilon t)$ (equation (13)).

The integration over ω' and then over ω yields

$$\psi_0 = \pm\mathrm{i}\int_{\omega_0-\omega_1}^{\omega_0+\omega_1} \frac{\varepsilon\sqrt{\omega_1^2-(\omega-\omega_0)^2}}{4\omega^3}\,\mathrm{d}\omega = \pm\frac{\mathrm{i}\varepsilon\pi}{8}\frac{\omega_1^2}{\left(\omega_0^2-\omega_1^2\right)^{3/2}}. \tag{15}$$

Thus, during one period of $\omega(\varepsilon t)$, the asymptotic solutions to the oscillator equation gain a nonzero generalized geometric phase (opposite in sign for two independent solutions) in addition to the usual dynamic phase. Let us recall that the terms in the square brackets in equation (9) are equal to zero under cyclic evolution, and the gain of the system phase under one-period variation $\omega(\varepsilon t)$ (13) is equal to

$$\varphi_0 = \pm\mathrm{i}\int_0^{2\pi\varepsilon^{-1}} \omega\,\mathrm{d}\tau + \psi_0 = \pm\mathrm{i}\left[2\pi\varepsilon^{-1}\omega_0 + \frac{\varepsilon\pi}{8}\frac{\omega_1^2}{\left(\omega_0^2-\omega_1^2\right)^{3/2}}\right] + O(\varepsilon^4 t). \tag{16}$$

At large times $t \gg \varepsilon^{-2}$, the generalized geometric phase increases infinitely, which is a direct consequence of its nonlocality (see [1–3]). Indeed, continuous circulation along a bounded contour can grow infinitely for a nonpotential field only. In this case, an increment of the generalized geometric phase is approximately equal to

$$\psi \cong \psi_0\left[\frac{\varepsilon t}{2\pi} + O(1)\right]. \tag{17}$$

Since $\psi_0 \sim \varepsilon$, the asymptotic solutions (8) and (9) can be written as

$$x = x(0)\sqrt{\frac{\omega(0)}{\omega(\varepsilon t)}}\exp\left\{\pm\mathrm{i}\int_0^t \omega\,\mathrm{d}\tau + \frac{\varepsilon\psi_0}{2\pi}t + O(\varepsilon)\right\} + O(\varepsilon^4 t). \tag{18}$$

Note that solution (18) represents, in essence, oscillations with a varying amplitude and an efficient (mean) frequency

$$\omega^{\mathrm{eff}} = \langle\omega\rangle_t + \frac{\varepsilon}{2\pi}|\psi_0|. \tag{19}$$

Here $\langle\omega\rangle_t = \frac{1}{t}\int_0^t \omega(\varepsilon\tau)\,\mathrm{d}\tau$ is the mean value of the current eigenfrequency. Thus, as indicated in [1–3], geometric phases cause a shift of efficient eigenfrequencies of a system. In our case, the efficient eigenfrequency is growing.

4. Conclusion

In this paper, the asymptotic solutions to the equation of an adiabatic harmonic oscillator, involving terms of order ε^3, have been obtained and investigated. We have analysed the geometric representation of solutions in a generalized parameter space, whose general theory has been elaborated in [3]. It has been shown that the terms of order ε^2 are nonlocal and give rise to a generalized geometric phase of solutions, which is described similarly to Berry's phase or Hannay's angle, but in the plane (ω, ω'). The real terms of orders ε and ε^3 are local and do not produce generalized geometric amplitudes. This result has been proved in general form in [3] and is closely related to the strong stability and the quantizability of Hamiltonian oscillatory systems.

We have calculated the increment of generalized geometric phase for one particular case. The simplest periodic time dependence of the single parameter—oscillator eigenfrequency—would suffice to give rise to this phase. In this respect, the phenomenon is less 'exotic' than Berry's geometric phase or Hannay's angle, when independent variations of several real parameters are necessary for its initiation in the case of linear ordinary differential equations. It has been demonstrated that owing to its nonlocality, the generalized geometric phase can increase infinitely even under small limited variations of ω (for example, periodical). This causes an efficient shift of the mean oscillator frequency over large times.

Note also that the asymptotic solutions obtained are applicable over times $t \ll \varepsilon^{-4}$ as long as the remainder term in solutions (2), (3), (8) and (9) is small. Considering that the generalized geometric phase (16) is of order $\varepsilon^2 t$, the above expressions adequately describe the solution behaviour up to very large values of the phase increment.

Since the equation for a harmonic oscillator is fundamental for many physical systems, the problem discussed above is applicable both for studying mechanical oscillators or electromagnetic contours and for analysing wave propagation in one-dimensionally inhomogeneous media or behaviour of quantum particles in an external potential.

References

[1] Bliokh K Yu 2002 *J. Math. Phys.* **43** 25

[2] Bliokh K Yu 2001 *Izv. Vuzov: Appl. Nonlinear Dyn.* **9** 45

[3] Bliokh K Yu 2002 *J. Math. Phys.* **43** 5624

[4] Berry M V 1984 *Proc. R. Soc.* A **392** 45

[5] Hannay J H 1985 *J. Phys. A: Math. Gen.* **18** 221

[6] Shapere A and Wilczek F (ed) 1989 *Geometric Phases in Physics* (Singapore: World Scientific)

[7] Vinitskiy S I, Debrov V L, Dubovik V M, Markovski B L and Stepanovskiy Yu P 1990 *Usp. Fiz. Nauk* **160** 6 (in Russian)

[8] Bender C M and Orszag S A 1978 *Advanced Mathematical Methods for Scientists and Engineers* (New York: McGraw-Hill)